m-Hypergeometric Solutions of Linear Recurrence and q-Recurrence Equations

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Abstract

In Koepf (1995), Koepf presents an algorithm to find an m-hypergeometric solution s_n of

$$s_{n+m} - s_n = a_n,$$

where a_n is a given m-hypergeometric term. We give a q-analogue of that algorithm. Also we generalize Koepf's algorithm to find m-hypergeometric solutions of linear recurrence equations without any restriction on the coefficients. Then we solve the same problem for linear q-recurrence equations.

Keywords **:** Gosper algorithm, m-hypergeometric solution, q-Gosper algorithm, qm-hypergeometric solution.

الحلول الهايبرجيومترية-m للمعادلات التكرارية والمعادلات التكرارية-q

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الخلاصة

في كوف (1995)، قدم كوف خوارزمية لايجاد الحلول الهايبرجيومترية -m s_n للمعادلة

$$s_{n+m} - s_n = a_n,$$

حيث ان a_n هو حد هايبرجيومتري معطى. نعطي الصيغة النظيرة -q لهذه الخوارزمية. كذالك نعمم خوارزمية كوف لايجاد الحلول الهايبرجيومترية -m المعادلات التكرارية الخطية بدون اي قيود على المعاملات. ثم نحل نفس المسألة للمعدلات التكرارية -q الخطية.

1. Introduction

Let m denotes a positive integer, $\mathbb{N}$ be the set of natural numbers, K be the field of characteristic zero, $K(n)$ be the field of rational functions over K, $K[n]$ be the ring of polynomials over K, and F denotes the transcendental extension of K by the indeterminate q, i.e., $F = K(q)$. In this paper, we will use x as an abbreviation for q^k. Recall that a non-zero term t_n is called a hypergeometric over K if there exist a rational function $r \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper's algorithm (Gosper, 1978) (see also Graham *et al.*, 1989, Koepf, 1998, Petkovšek *et al.*, 1996) has been extensively studied and widely used to prove hypergeometric identities. Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \tag{1.1}$$

if it exists, or confirm the nonexistence of any solution of (1.1). A non-zero term a_n is called an m-hypergeometric over K if there exist a rational function $w(n) \in K(n)$ such that.

$$\frac{a_{n+m}}{a_n} = w(n),$$

In Koepf (1995), Koepf extends Gosper's algorithm to find m-hypergeometric solution s_n of

$$s_{n+m} - s_n = a_n, \tag{1.2}$$

where a_n is a given m-hypergeometric term. In Petkovšek and Bruno (1993), Petkovšek and Bruno described an algorithm to find m-hypergeometric solutions of the linear recurrence equation

$$\sum_{i=0}^{d} p_i s_{n+mi} = 0, \tag{1.3}$$

where d is a positive integer and $\{p_i(n)\}_{i=0}^{d}$ are given polynomials over K. Their algorithm reduces to algorithm **Hyper** (Petkovšek, 1992) when $m = 1$. Recall that a non-zero term h_k is called a q-hypergeometric over F if there exist a rational function $\sigma \in F(x)$ such that

$$\frac{h_{k+1}}{h_k} = \sigma(x).$$

q-Gosper's algorithm (see Böing and Koepf, 1999, Koornwinder, 1993, Paule and Riese, 1997, Paule and Strehl, 1995) determines if there exists a q-hypergeometric term anti-difference of a given q-hypergeometric term and computes this anti-difference provided that it exists. A non-zero term f_k is called a qm-hypergeometric over F if there exist a rational function $\rho \in F(x)$ such that

$$\frac{f_{k+m}}{f_k} = \rho(x).$$

The contents of this paper are as follows: In Section 2, we give a q-analogue of Koepf's algorithm. In Section 3, we show that Koepf's algorithm can be generalized to find m-hypergeometric solutions of non-homogenous linear recurrence equations without any restrictions on the coefficients. Finally, in Section 4, we solve the same problem in Section 3 for the q-case.

2. qm-Hypergeometric Solutions of Anti-Recurrence Equations

In this section, we give a q-analogue of Koepf's algorithm, i.e., find a qm-hypergeometric solution g_k of

$$g_{k+m} - g_k = f_k, \tag{2.1}$$

where f_k is a given qm-hypergeometric term.

Theorem 2.1. *Given a qm-hypergeometric term f_k. If the equation*

$$h_{k+1} - h_k = f_{km}, \tag{2.2}$$

has a q-hypergeometric solution with respect to h_k, then equation (2.1) *has a qm-hypergeometric solution with respect to g_k given by $g_k = h_{k/m}$, otherwise equation* (2.1) *has no qm-hypergeometric solution.*

Proof. If g_k is a qm-hypergeometric solution of equation (2.1), then by using (2.1), we find

$$\frac{g_k}{f_k} = \frac{g_k}{g_{k+m} - g_k} = \frac{1}{\dfrac{g_{k+m}}{g_k} - 1}.$$

Let $\tau(x) = g_k \big/ f_k$. It follows that $\tau(x)$ is a rational function of x. Substituting $\tau(x) f_k$ for g_k in (2.1) to obtain

$$\rho(x)\tau(q^k m) - \tau(x) = 1, \tag{2.3}$$

where $\rho(x) = f_{k+m} \big/ f_k$ is a rational function of x. Analogously, from (2.2) we get

$$\rho(x^m)\mu(qx) - \mu(x) = 1, \tag{2.4}$$

where $\mu(x) = h_k \big/ f_{km}$ is a rational function of x. Clearly (2.3) and (2.4) are either both solvable and have solutions such that $\mu(x) = \tau(x^m)$, or are both unsolvable. So either (2.2) has no q-hypergeometric solution and (2.1) has no qm-hypergeometric solution, or (2.2) has a q-hypergeometric solution $h_k = \mu(x) f_{km}$ and (2.1) has the desired solution. □

Algorithm 2.1.

INPUT : *a* qm*-hypergeometric term* f_k.

OUTPUT : *a* qm*-hypergeometric solution* g_k *of* (2.1) *if it exists, otherwise* "*no* qm*-hypergeometric solution of* (2.1) *exists*".

(1) *Compute the* q*-hypergeometric solution* h_k *of equation* (2.2) *if it exists, otherwis return* "*no* qm*-hypergeometric solution of* (2.1) *exists*".

(2) *Compute the* qm*-hypergeometric solution* g_k *of equation* (2.1) *by the following relation:*

$$g_k = h_{k/m}.$$

3. m-Hypergeometric Solutions of Linear Recurrence Equations

In this section, we generalize Koepf's algorithm to find m-hypergeometric solutions of the linear recurrence equation

$$\sum_{i=0}^{d} p_i(n) s_{n+mi} = a_n, \tag{3.1}$$

where $\{p_i(n)\}_{i=0}^{d}$ are given polynomials and a_n is a given m-hypergeometric term.

Theorem 3.1. *Given an* m *-hypergeometric term* a_n. *If the equation*

$$\sum_{i=0}^{d} p_i(mn)t_{n+i} = a_{mn} , \tag{3.2}$$

has a hypergeometric solution with respect to t_n, *then equation* (3.1) *has an* m *-hypergeometric solution with respect to* s_n *given by* $s_n = t_{n/m}$, *otherwise equation* (3.1) *has no* m *-hypergeometric solution.*

Proof. If s_n is an m -hypergeometric solution of equation (3.1), then the left hand-side of (3.1) can
be written as a rational function multiple of s_n. Let $S(n) = s_n / a_n$. It follows that $S(n)$ is a rational

function of n. Substituting $S(n)a_n$ for s_n in (3.1) to obtain

$$\sum_{i=0}^{d} p_i(n)S(n+mi)\prod_{j=0}^{i-1} w(n+mj) = 1, \tag{3.3}$$

where $w(n) = a_{n+m} / a_n$ is a rational function of n. Analogously, from (3.2) we get

$$\sum_{i=0}^{d} p_i(mn)T(n+i)\prod_{j=0}^{i-1} w(m(n+i)) = 1, \tag{3.4}$$

where $T(n) = t_n / a_{mn}$ is a rational function of n. Clearly (3.3) and (3.4) are either both solvable and have solutions such that $T(n) = S(mn)$, or are both unsolvable. So either (3.2) has no hypergeometric solution and (3.1) has no m -hypergeometric solution, or (3.2) has a hypergeometric solution $t_n = T(n)a_{mn}$ and (3.1) has the desired solution. □

Algorithm 3.1.
INPUT : $\{p_i(n)\}_{i=0}^{d} \in K[n]$ *and an m-hypergeometric term* a_n.
OUTPUT : *an* m *-hypergeometric solution* s_n *of* (3.1), *if it exists, otherwise "no* m *-hypereometric solution of* (3.1) *exists".*

(1) *Compute the hypergeometric solution* t_n *of equation* (3.2) *if it exists, otherwise return "no* m *-hypergeometric solution of* (3.1) *exists".*

(2) *Compute the* m*-hypergeometric solution* s_n *of equation* (3.1) *by the following relation*:

$$s_n = t_{n/m}.$$

Example 3.1. *We want to find all* 2*-hypergeometric solutions of*

$$s_{n+4} + (n^3 + 15n^2 + 3)s_{n+2} - n(n+5)(n+10)s_n = 4 - 50n. \tag{3.5}$$

By equation (3.2), t_n *is a hypergeometri term, which satisfies*

$$t_{n+2} + (8n^3 + 60n^2 + 3)t_{n+1} - 2n(2n+5)(2n+10)t_n = 4 - 100n.$$

The only hypergeometric solution of this equation is $t_n = 1$. *Thus* $s_n = t_{n/2} = 1$ *is the only* 2 *-hypergeometric solution of* (3.5). □

4. *qm*-Hypergeometric Solutions of Linear *q*-Recurrence Equations

We present an algorithm to find a qm-hypergeometric term g_k satisfying

$$\sum_{i=0}^{d} \lambda_i(x) g_{k+mi} = f_k, \tag{4.1}$$

where $\{\lambda_i(x)\}_{i=0}^{d}$ are given polynomials and f_k is a given qm-hypergeometric term. This algorithm is a generalization of the one given in Section 2. Also, it is a q-analogue of the one given in Section 3.

Theorem 4.1. *Given a* qm*-hypergeometric term* f_k. *If the equation*

$$\sum_{i=0}^{d} \lambda_i(x^m) h_{k+i} = f_{km}, \tag{4.2}$$

has a q*-hypergeometric solution with respect to* h_k, *then equation* (4.1) *has a* qm*-hypergeometric solution with respect to* g_k *given by* $g_k = h_{k/m}$, *otherwise equation* (4.1) *has no* qm*-hypergeometric solution.*

Proof. Let $\tau(x), \rho(x), \mu(x)$ be defined as in Section 2. It follows that $\rho(x)$ is a rational function of x. If g_k is a qm-hypergeometric solution of equation (4.1), then the left hand-side of (4.1) can be written as a rational function multiple of g_k. It follows that $\tau(x)$ is a rational function of x. Substituting $\tau(x) f_k$ for g_k in (4.1) to obtain

$$\sum_{i=0}^{d} \lambda_i(x)\tau(q^{mi}x)\prod_{j=0}^{i-1}\rho(q^{mj}x) = 1. \tag{4.3}$$

Analogously, from (4.2) we get

$$\sum_{i=0}^{d} \lambda_i(x^m)\mu(q^i x)\prod_{j=0}^{i-1}\rho(q^{mj}x^m) = 1, \tag{4.4}$$

where $\mu(x)$ is a rational function of x. Clearly (4.3) and (4.4) are either both solvable and have
solutions such that $\mu(x) = \tau(x^m)$, or are both unsolvable. So either (4.2) has no q-hypergeometric solution and (4.1) has no qm-hypergeometric solution, or (4.2) has a q-hypergeometric solution $h_k = \mu(x) f_{km}$ and (4.1) has the desired solution. □

Algorithm 4.1.
INPUT : $\{\lambda_i(x)\}_{i=0}^{d} \in F[x]$ *and a* qm*-hypergeometric term* f_k.
OUTPUT: *a* qm*-hypergeometric solution* g_k *of* (4.1) *if it exists, otherwise* "*no* qm*-hypergeometric solution of* (4.1) *exists*".

(1) *Compute the* q*-hypergeometric solution* h_k *of equation* (4.2) *if it exists, otherwise return* "*no* qm*-hypergeometric solution of* (4.1) *exists*".

(2) *Compute the* qm*-hypergeometric solution* g_k *of equation* (4.1) *by the following relation*:

$$g_k = h_{k/m}.$$

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